Artificial Intelligence 2 – SS 2020
Assignment 1: Introduction to Probability
– Given April 24., Due May 03. –

Some Basic Notions Assume we have a set of elementary events Ω.

- The expression $P(A|B)$ means "The probability of A *given* B" (conditional probabilities), i.e. the probability of event A under the assumption that the event B has occured.

  We can define this as $P(A|B) = \frac{P(A \wedge B)}{P(B)}$.

- One of the central tools in probability theory is Bayes' theorem:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

- Bayes' theorem is a formal theorem telling us how to *update our beliefs* in a *hypothesis* when confronted with new *evidence*. Let $\mathcal{H}$ the "space of all competing hypotheses", $H \in \mathcal{H}$ and e a new piece of evidence. Then Bayes theorem tells us, that

$$P(H|e) = \frac{P(e|H) \cdot P(H)}{P(e)}$$

  We call $P(H)$ the *prior* (or *a priori* probability); the probability you *thought* H was true before encountering e. We call $P(e|H)$ the *likelihood*; the probability with which your hypothesis H *predicted* the evidence e (before encountering it). We call $P(H|e)$ the *a posteriori* probability that H is true.

  Note, that $P(e)$ can be decomposed as $P(e) = P(e|H) \cdot P(H) + P(e|\neg H) \cdot P(\neg H) = \sum_{h \in \mathcal{H}} P(e|h) \cdot P(h)$. In other words, the probability of encountering a piece of evidence e *at all* is the sum of individual priors$\cdot$likelihoods over all competing hypotheses.

- To update my belief in a hypothesis H given evidence e is to compute the a posteriori probability of H using Bayes' theorem, and to accept the result as my new prior going forward.

Axiom of Bayesian Rationality: Whenever you learn from evidence, the degree to which you learn *correctly* is precisely the degree to which your update of subjective probabilities corresponds to the result of applying Bayes' theorem.

Here are two warm up problems, to get you started thinking with conditional probabilities.

Problem 1.1 (Disjunctive Random Variables)

We know that given boolean random variables A and B we have 20pt

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

Extend this formula to the case of three random variables $P(A \vee B \vee C)$. Draw a Venn diagram to "prove" your formula.

Problem 1.2 (Monty Hall Problem)

The Monty Hall problem is a famous example of how our intuition (even that of highly trained mathematicians) can fail when it comes to estimating probabilities. 30pt

> Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

The following steps guide you through one of the possible scenarios. Try to take note of the assumptions each step makes (including some implicit ones)—it is good practice. You can read more about the problem, including a discussion of the various solutions, on Wikipedia: `https://en.wikipedia.org/wiki/Monty_Hall_problem`. The article is an interesting read!

Remember, it is easier for us to give points for answers that are not completely correct if you explain your thinking.

1. Assume that it is equally probable that the car is behind door No. 1, No. 2, or No. 3. What is the probability of the event A_1 of the car being behind No. 1? 3 pt

2. You pick door No. 1. What are the probabilities of events B_1, B_2, and B_3 of the car being behind door No. 1, No. 2, or No. 3 after you have picked door No. 1? 3 pt

3. Assume that Monty always opens a door that you did not pick and that has a goat. This means that if you picked a door with a goat, he will always pick the other door with a goat. If he has two doors to choose from (if you picked the door with a car), the probabilities of him choosing either one are equal. 6 pt

   What is the probability of the event G_2 that Monty shows you a goat behind door No. 2, after we have chosen door No. 1?

4. Compute the combined probabilities $P(G_j C_i)$ of Monty showing you a goat behind door No. $j \in 2, 3$ *and* the car being behind door No. $i \in 1, 2, 3$. 8 pt

5. What is the *conditional* probability $P(G_2|C_1)$ of the event that Monty shows you a goat behind door No. 2 given that the car is behind door No. 1. **Write down the calculation, no point will be given for just the number.** 4 pt

6. Finally, using Bayes' theorem, calculate the conditional probability $P(C_1|G_2)$. **Write down the calculation, no point will be given for just the number.** 6 pt