Klausur C2 vom 15.06.2020

– Lösungsvorschlag –

A1

a) $$\lim_{x\to 0} \frac{\overbrace{e^{-\alpha x}}^{\to e^0=1} - 1 + \overbrace{\alpha x}^{\to 0}}{\underbrace{e^{\beta x}}_{\to e^0=1} - 1 + \underbrace{\beta x}_{\to 0}} \underset{\text{Hosp.}}{=} \lim_{x\to 0} \frac{-\alpha e^{-\alpha x} + \alpha}{\beta e^{\beta x} - \beta}$$

$$= \frac{-\alpha}{\beta} \cdot \lim_{x\to 0} \frac{\overbrace{e^{-\alpha x} - 1}^{\to 0}}{\underbrace{e^{\beta x} - 1}_{\to 0}} \underset{\text{Hosp.}}{=} \frac{-\alpha}{\beta} \cdot \lim \frac{-\alpha \cdot \overbrace{e^{-\alpha x}}^{\to 1}}{\beta \cdot \underbrace{e^{\beta x}}_{\to 1}}$$

$$= \left(-\frac{\alpha}{\beta}\right)^2 = \frac{\alpha^2}{\beta^2}$$

b) $$\lim_{x\to 0} \frac{e^{\overbrace{1-\cos x}^{\to 0}} - 1}{\underbrace{x^2}_{\to 0}} \underset{\substack{\text{Typ „}\frac{0}{0}\text{"} \\ \text{Hospital}}}{=} \lim_{x\to 0} \frac{e^{1-\cos x} \cdot \sin x}{2x}$$

$$= \frac{1}{2} \cdot \lim_{x\to 0} e^{1-\cos x} \cdot \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2} \cdot e^0 \cdot 1 = \frac{1}{2}$$

Alternative: nochmal Hospital:

$$\ldots = \lim_{x\to 0} \frac{e^{1-\cos x} \cdot \sin^2 x + e^{1-\cos x} \cdot \cos x}{2} = \frac{e^0 \cdot 0 + e^0 \cdot 1}{2} = \frac{1}{2}$$

c) $$\lim_{x \to 0} \frac{\overbrace{\alpha\sqrt{x}}^{\to 0} + \overbrace{\frac{\beta}{x}}^{\to +\infty}}{\underbrace{\gamma\sqrt{x}}_{\to 0} + \underbrace{\frac{\delta}{x}}_{\to +\infty}}$$ Typus „$\frac{\infty}{\infty}$"

Die „dominante" x-Potenz (die, die das 'Problem' verursacht) ausklammern und durch Kürzen:

$$= \lim_{x \to 0} \frac{\cancel{\frac{1}{x}} \cdot \left(\alpha x^{3/2} + \beta\right)}{\cancel{\frac{1}{x}} \cdot \left(\gamma x^{3/2} + \delta\right)} = \lim_{x \to 0} \frac{\alpha \overbrace{x^{3/2}}^{\to 0} + \beta}{\gamma \underbrace{x^{3/2}}_{\to 0} + \delta} = \underline{\underline{\frac{\beta}{\delta}}}$$

<u>Bem.</u>: Regel von l'Hospital nützt hier nichts:

$$\lim_{x \to 0} \frac{\alpha\sqrt{x} + \frac{\beta}{x}}{\gamma\sqrt{x} + \frac{\delta}{x}} = \lim_{x \to 0} \frac{\frac{\alpha}{2\sqrt{x}} - \frac{\beta}{x^2}}{\frac{\gamma}{2\sqrt{x}} - \frac{\delta}{x^2}} \underset{\text{mit } x \text{ erweitern?}}{=} \lim_{x \to 0} \frac{\frac{\alpha}{2}\sqrt{x} - \frac{\beta}{x}}{\frac{\gamma}{2}\sqrt{x} - \frac{\delta}{x}} = ?$$

A2

a) (ii) $\lim\limits_{n\to\infty} \frac{\alpha\sqrt{n} + \frac{\beta}{n}}{\gamma\sqrt{n} + \frac{\delta}{n}}$ Typus „$\frac{\infty}{\infty}$"

höchsten Potenzen rausziehen und durch Kürzen (denn diese verursachen den Typus)

$$= \lim\limits_{n\to\infty} \frac{\cancel{\sqrt{n}}\left(\alpha + \frac{\beta}{n^{3/2}}\right)}{\cancel{\sqrt{n}}\left(\gamma + \frac{\delta}{n^{3/2}}\right)} = \lim\limits_{n\to\infty} \frac{\alpha + \frac{\beta}{n^{3/2}}}{\gamma + \frac{\delta}{n^{3/2}}} = \underline{\underline{\frac{\alpha}{\gamma}}}$$

(ii) $\lim\limits_{n\to\infty} \left(1 - \frac{3}{nx}\right)^{4n} = \left(\lim\limits_{n\to\infty} \left(1 + \frac{-3/x}{n}\right)^{n}\right)^{4}$

$$\underset{\text{Vorl.}}{=} \left(e^{-3/x}\right)^4 = \underline{\underline{e^{-12/x}}}$$

b) $\sqrt[k]{|a_k|} = 3\frac{\sqrt[k]{2}}{\sqrt[k]{k}} + \frac{5}{\sqrt[k]{k!}} \xrightarrow{(k\to\infty)} 3 \Rightarrow \underline{\underline{R = \frac{1}{3}}}$

$\sqrt[k]{2} \to 1$, $\sqrt[k]{k} \to 1$, $\sqrt[k]{k!} \to +\infty$

$\boxed{A3}$ $f(x) = \sin(x) \cdot e^{\frac{2x}{\pi}} + \frac{2\cos x}{\pi} - 2$, $I = (0, \frac{\pi}{2})$

Teil 1:

$$f(0) = 0 \cdot e^0 + \frac{2 \cdot 1}{\pi} - 2 = \underbrace{\frac{2}{\pi}}_{<1} - 2 < -1 < 0$$

$$f(\tfrac{\pi}{2}) = 1 \cdot e^1 + \frac{2 \cdot 0}{\pi} - 2 = \underbrace{e}_{>2} - 2 > 0$$

f stetig

$\Longrightarrow$ f hat auf I mindestens eine Nullstelle

(Nullstellensatz oder Zwischenwertsatz von Bolzano)

Teil 2:

$$f'(x) = \cos(x) \cdot e^{\frac{2x}{\pi}} + \frac{2}{\pi} \sin(x) \cdot e^{\frac{2x}{\pi}} - \frac{2}{\pi} \sin(x)$$

$$= \underbrace{\cos(x)}_{>0} \cdot \underbrace{e^{\frac{2x}{\pi}}}_{>0} + \frac{2}{\pi} \cdot \underbrace{\sin(x)}_{>0} \cdot \underbrace{(\underbrace{e^{\overset{>0}{\frac{2x}{\pi}}}}_{>1} - 1)}_{>0} > 0 \ \forall x \in I$$

$\Longrightarrow$ f ist auf I streng monoton wachsend

$\Longrightarrow$ f hat auf I höchstens eine Nullstelle

$\boxed{A4}$ a) $\displaystyle\int_0^{1/2} 6\cdot(2x+1)^{-4}\,dx$

$y := 2x+1$

$\rightsquigarrow \frac{dy}{dx} = 2 \rightsquigarrow 2\,dx = dy$

$x=0 \rightsquigarrow y=1$

$x=\frac{1}{2} \rightsquigarrow y=2$

$$= \int_1^2 3\cdot y^{-4}\,dy$$

$$= 3\cdot\left(-\tfrac{1}{3}\right)y^{-3}\Big|_1^2 = -\left(\tfrac{1}{8}-1\right) = \underline{\underline{\tfrac{7}{8}}}$$

b) $\displaystyle\int \frac{\sqrt{1+\ln x}}{x}\,dx$

$y := \ln x$

$\rightsquigarrow \frac{dy}{dx} = \frac{1}{x} \rightsquigarrow \frac{dx}{x} = dy$

$$= \int (1+y)^{1/2}\,dy = \frac{2}{3}(1+y)^{3/2} = \underline{\underline{\frac{2}{3}(1+\ln x)^{3/2}}} \quad (+c)$$

c) $\displaystyle\int_{1/\pi}^{2/\pi} \frac{\cos\left(\frac{1}{x}\right)}{x^3}\,dx$

$y := \frac{1}{x} \rightsquigarrow \frac{dy}{dx} = -\frac{1}{x^2} \rightsquigarrow \frac{dx}{x^2} = -dy$

$$= -\int_{\pi}^{\pi/2} y\cdot\cos y\,dy = \int_{\pi/2}^{\pi} \underbrace{y}_{\text{abl.}}\cdot\overbrace{\cos y}^{\text{int.}}\,dy$$

$$\overset{\text{p.I.}}{=} y\cdot\sin y\Big|_{\frac{\pi}{2}}^{\pi} - \int_{\pi/2}^{\pi} 1\cdot\sin y\,dy$$

$$= \pi\cdot\underbrace{\sin\pi}_{=0} - \frac{\pi}{2}\cdot\underbrace{\sin\frac{\pi}{2}}_{=1} + \cos y\Big|_{\frac{\pi}{2}}^{\pi} = -\frac{\pi}{2} + (-1) + 0$$

$$= \underline{\underline{-1-\frac{\pi}{2}}}$$

d)

$$\int \underbrace{\sin(nx)}_{\text{abl.}} \cdot \underbrace{e^x}_{\text{int}} \, dx$$

$$\underset{\text{p.I.}}{=} \sin(nx) \cdot e^x - \int n \underbrace{\cos(nx)}_{\text{abl.}} \cdot \overbrace{e^x}^{\text{int.}} \, dx$$

$$\underset{\text{p.I.}}{=} \sin(nx) \cdot e^x - n \cdot \left[\cos(nx) \cdot e^x - \int (-n \cos(nx)) \cdot e^x \, dx \right]$$

$$= \sin(nx) \cdot e^x - n \cdot \cos(nx) \cdot e^x - n^2 \int \cos(nx) \cdot e^x \, dx$$

$$\Big| + n^2 \int \cos(nx) \cdot e^x \, dx$$

$$\Longrightarrow (1+n^2) \cdot \int \sin(nx) \cdot e^x \, dx = [\sin(nx) - n \cdot \cos(nx)] \cdot e^x$$

$$\Longrightarrow \int \sin(nx) \cdot e^x \, dx = \frac{[\sin(nx) - n \cdot \cos(nx)] \cdot e^x}{1+n^2} \quad (+c)$$

A5

a) $f(x) = x \cdot (\ln x)^2$

$f'(x) = 1 \cdot (\ln x)^2 + x \cdot 2 \ln x \cdot \frac{1}{x} = (\ln x)^2 + 2 \ln x$

$f''(x) = 2 \ln x \cdot \frac{1}{x} + \frac{2}{x} = \frac{2}{x} \cdot (1 + \ln x)$

$f(e) = e \cdot 1^2 = e$

$f'(e) = 1^2 + 1 \cdot 2 = 3$

$f''(e) = \frac{2}{e} \cdot (1+1) = \frac{4}{e}$

$$T_2(x) = f(e) + f'(e) \cdot (x-e) + \frac{f''(e)}{2} \cdot (x-e)^2$$

$$= \underline{\underline{e + 3 \cdot (x-e) + \frac{2}{e} \cdot (x-e)^2}}$$

b) $f'''(x) = -\frac{2}{x^2} \cdot (1 + \ln x) + \frac{2}{x} \cdot \frac{1}{x} = -\frac{2 \ln x}{x^2}$

$R_2(x) = \frac{f'''(\xi)}{3!} \cdot (x-e)^3 = -\frac{\cancel{2} \ln \xi}{\cancel{2} \cdot 3 \xi^2} \cdot (x-e)^3$ wobei ξ zw. x u. e

c) $R_2(2e) = -\frac{\ln \xi}{3\xi^2} \cdot (2e-e)^3 = -\frac{\ln \xi}{3\xi^2} \cdot e^3$ wobei $e < \xi < 2e$

$\Downarrow$

$\ln \xi < \ln(2e)$

$\frac{1}{\xi^2} < \frac{1}{e^2}$

$\rightsquigarrow |R_2(2e)| = \frac{\overset{< 2e}{\ln \xi}}{3 \underset{> e^2}{\xi^2}} \cdot e^3$

$< \frac{\ln(2e)}{3e^2} \cdot e^3 = \frac{\ln(2e) \cdot e}{3}$

$= \underline{\underline{\frac{(1 + \ln 2) \cdot e}{3}}}$ $\quad \underline{\left[< \frac{2}{3} e \right]}$

d) $\tilde{x}_* = e^2$

$\tilde{y}_* = f(\tilde{x}_*) = f(e^2) = e^2 \cdot 2^2 = 4e^2$

$$\tilde{T}_1(y) = \underbrace{(f^{-1})(\tilde{y}_*)}_{= \tilde{x}_*} + \underbrace{(f^{-1})'(\tilde{y}_*)}_{= \frac{1}{f'(x_*)}} \cdot (y - \tilde{y}_*)$$

$f'(e^2) = 2^2 + 2 \cdot 2 = 8$

$$\Rightarrow \quad \underline{\underline{\tilde{T}_1(y) = e^2 + \frac{1}{8} \cdot (y - 4e^2)}}$$